# DETERMINATION OF FORCES AND MOMENTS DUE TO LIGHT PRESSURE ACTING ON A BODY IN MOTION IN COSMIC SPACE

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One of the factors which is influencing the motion of any object in the near-solar space is the pressure force of solar radiation. In recent years a number of papers were published [1-5] where the authors analyzed systems which utilize the solar radiation pressure effect either for translating the center of mass of the satellite (solar sail) or for creating moments which control the angular orientation of the satellite (solar control). In these papers it was usually assumed that the light pressure acts on a plane which is part of the driving assembly or part of the attitude control system of a space vehicle. In this paper, using a more general description as a basis, it is attempted to obtain integral characteristics for the determination of the action of the light flow on the body of a space vehicle.

If in a vacuum a light flux propagates in a given direction, and its energy entering a unit volume is equal to $w$, then the momentum, corresponding to the unit volume, is equal to $K$

$$K = \frac{w}{c} \tag{1}$$

where $c$ is the speed of light in vacuum. The momentum vector $\mathbf{K}$ is pointed in the direction of propagation of the light. During the interaction of the light with any body there occurs a change of the momentum vector, as a result of which the body is subjected to an impulse $\mathbf{f}\Delta t$, which is related to the change of momentum $\Delta\mathbf{K}$ by the usual relation

$$\mathbf{f}\Delta t = \Delta\mathbf{K} \tag{2}$$

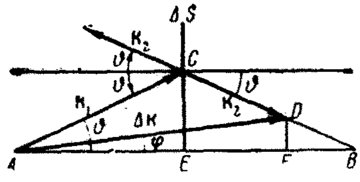


Fig. 1.

where $\Delta t$ is the time during which the momentum vector changes by $\Delta\mathbf{K}$.

Let us study the interaction of the light with an area element $\Delta S$ (Fig. 1). There we shall assume that the incidence angle of the light flux is equal to the reflection angle, the incident and the reflected fluxes, as well as the perpendicular to the surface element lie in one plane; the amount of energy in a unit volume of reflected flux represents a part of the energy contained in a unit volume of the incident light flux. Let us denote by $\vartheta$ the incidence angle of the light flow, $\mathbf{K}_1$ and $\mathbf{K}_2$ are the momentum vectors of the incident and reflected light flow, respectively. The magnitudes of the vectors $|\mathbf{K}_2|$ and $|\mathbf{K}_1|$ are related as follows:

$$|\mathbf{K}_2| = \varepsilon |\mathbf{K}_1| \tag{3}$$

where $\varepsilon$ is the reflection coefficient, i.e. the ratio of the energy density of the reflected light flux to the energy density of the incident light flux. With full reflection $\varepsilon = 1$, with full absorption $\varepsilon = 0$.

With the above assumptions we obtain from the triangle $ACD$

$$|\Delta\mathbf{K}|^2 = |\mathbf{K}_1|^2 + |\mathbf{K}_2|^2 - 2|\mathbf{K}_1||\mathbf{K}_2| \cos(180^\circ - 2\vartheta)$$

or, considering (3)

$$|\Delta\mathbf{K}| = |\mathbf{K}_1| \sqrt{1 + \varepsilon^2 + 2\varepsilon \cos 2\vartheta} \tag{4}$$

Furthermore, from the triangles $ACE$ and $ADF$ we obtain

$$CE = |\mathbf{K}_1| \sin\vartheta, \quad DF = |\Delta\mathbf{K}| \sin\varphi$$

Here $\varphi$ is the angle between the vector $\Delta\mathbf{K}$, and consequently, also the force $\mathbf{f}$, and the perpendicular to the surface element.

Since the triangles $DBF$ and $BCE$ are similar

$$\frac{DF}{CE} = \frac{DB}{CB} = \frac{|\mathbf{K}_1| - |\mathbf{K}_2|}{|\mathbf{K}_1|} = 1 - \varepsilon$$

consequently,

$$\sin\varphi = \frac{(1-\varepsilon)|\mathbf{K}_1|\sin\vartheta}{|\Delta\mathbf{K}|} = \frac{(1-\varepsilon)\sin\vartheta}{\sqrt{1+\varepsilon^2+2\varepsilon\cos 2\vartheta}} \tag{5}$$

Now we shall determine the magnitude of $|\mathbf{K}_1|$. This magnitude is equal to the motion of the light flux arriving in the volume of the parallelepiped with a base area $\Delta S$ and sides $c\Delta t$ long. Since the light stream makes an angle $\vartheta$, with the normal to the surface, the volume of this parallelepiped is equal to $\Delta S c \Delta t \cos\vartheta$. The motion arriving at this volume is equal to

$$|\mathbf{K}_1| = \frac{w}{c} c\Delta t \ \Delta S \cos\vartheta = w \ \Delta t \ \Delta S \cos\vartheta \tag{6}$$

From the relations (2), (4), (5) and (6) it follows that the force element of light pressure $d\mathbf{f}$, acting on a surface element $dS$ is equal in magnitude to

$$df = w\sqrt{1+\varepsilon^2+2\varepsilon\cos 2\vartheta}\cos\vartheta \, ds \tag{7}$$

and encloses an angle with the normal to the surface of

$$\varphi = \sin^{-1}\left[\frac{(1-\varepsilon)\sin\vartheta}{\sqrt{1+\varepsilon^2+2\varepsilon\cos 2\vartheta}}\right] \tag{8}$$

With total reflection $\varepsilon = 1$, $df = 2w\cos^2\vartheta\, dS$ and $\varphi = 0$, i.e. the force of the light pressure is directed normal to the surface element.

With total absorption $\varepsilon = 0$, $df = w\cos\vartheta\, dS$ and $\varphi = \vartheta$, i.e. the force of the light pressure is directed along the incoming light stream.

The light energy density $w$ is related to the flow of light energy $E$ by $E = wc$. In turn, the magnitude of the flux of light energy, arriving at a unit surface of the body is inversely proportional to the square of the distance from the light source. Thus we obtain finally

$$df = \frac{E_0}{c}\left(\frac{r_0}{r}\right)^2\sqrt{1+\varepsilon^2+2\varepsilon\cos 2\vartheta}\cos\vartheta\, dS \tag{9}$$

where $E_0$ is the magnitude of the flux of light energy arriving at a unit surface of the body placed at a distance $r_0$ from the light source.

The magnitude of the flux of the sun's light energy at the distance of the earth's orbit [6] is

$$E_c = 1200\,\frac{\text{Kcal}}{\text{m}^2\text{hr}} = 1.39\cdot 10^6\,\frac{\text{erg}}{\text{cm}^2\text{sec}}$$

$$\frac{E_c}{c} = 0.464\cdot 10^{-4}\,\frac{\text{erg}}{\text{cm}^3} = 4.64\cdot 10^{-5}\,\frac{\text{dyne}}{\text{cm}^2} = 4.72\cdot 10^{-8}\,\frac{\text{gram}}{\text{cm}^2}$$

Consequently, for $\varepsilon = 0$ and $\vartheta = 0$ a surface of 2120 $\text{m}^2$ (46 m square) is necessary to obtain a force of 1 gram.

As an example, let us study the influence of light radiation upon the motion relative to the center of mass of a space vehicle of the form of a right circular uniform cylinder of radius $R$ and length $l$. Let us assume that the diametral plane of the cylinder is perpendicular to the direction of light radiation. Also let us assume that the turning moment relative to the cylinder axis arises from the fact that one-half of the lateral surface of the cylinder is absolutely reflective and the other half is absorbent with a reflection coefficient $\varepsilon < 1$. With these assumptions the problem reduces to the plane case.

Figure 2 shows a cross-section of the cylinder perpendicular to its axis. Axes $\eta$ and $\zeta$ are inertial reference axes, while $x$ and $y$ are rotating together with the cylinder. Here we shall assume that the light propagates along the negative direction of the $\eta$-axis. Let the contour of separation of the lateral surface of the cylinder into two halves with different reflection properties lie in the plane $y = 0$ and the surface $y > 0$ be absorbing, while $y < 0$ is absolutely reflective.

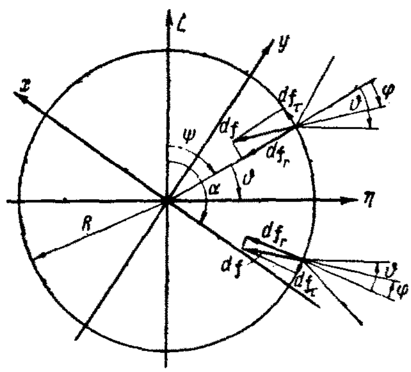

Fig. 2.

Then the equation of motion of the cylinder relative to its axis has the form

$$I\ddot{\alpha} + b_0 \sin^2\alpha \operatorname{sign} \sin\alpha = 0 \qquad (10)$$

$$\left(b_0 = \frac{1}{2}\frac{E_0}{c}\left(\frac{r_0}{r}\right)^2 R^2 l\,(1 - \varepsilon) = \text{const}\right)$$

Here $I$ is the moment of inertia of the cylinder. This equation is analogous to the equation of the mathematical pendulum with the difference that the restoring moment in the case at hand is proportional to the function $F(\alpha) = \sin^2\alpha \operatorname{sign} \sin\alpha$ instead of $F'(\alpha) = |\sin\alpha| \operatorname{sign} \sin\alpha = \sin\alpha$ as in the case of the mathematical pendulum.

The equation of the phase trajectory of motion, described by Equation (10) has the form

$$\frac{\dot{\alpha}^2}{2} + \Pi(\alpha) = C \qquad \left(\Pi(\alpha) = \omega_0^2 \int \sin^2\alpha \operatorname{sign}\sin\alpha\, d\alpha,\ \omega_0^2 = \frac{b}{I}\right) \qquad (11)$$

Here $\Pi(\alpha)$ is the potential energy of the cylinder, $C$ is a constant of integration.

Figure 3 shows a sample sketch of the functions $\sin^2\alpha$ sign $\sin\alpha$, $\Pi(\alpha)$ and a sample drawing of the phase trajectories. Thus, with a sufficiently small initial angular velocity the cylinder will execute undamped oscillations about a position of stable equilibrium which corresponds to a situation where the ideally reflective surface is turned toward the light. The maximum initial angular velocity of the cylinder in this region is determined as a point of the separatrix ($\alpha^0_{max}$, $\alpha^0 = 0$). From the equation of the separatrix it follows that

$$|\dot{\alpha}^{\circ}_{\max}| = \omega_0 \sqrt{\pi} \tag{12}$$

After substitution of (12) into the expression for the moment of inertia of the cylinder we obtain

$$|\dot{\alpha}^{\circ}_{\max}| = \frac{1}{R}\sqrt{\frac{h_0(1-\varepsilon)}{\gamma}} \qquad \left(h_0 = \frac{E_0}{c}\left(\frac{r_0}{r}\right)^2\right)$$

where $\gamma$ is the material density of the cylinder.

Let us assume that the source of light is the sun and the distance from the sun to the cylinder is equal to the distance between the sun and earth. Then $h_0 = 4.64 \times 10^{-5}$ dyne/cm$^2$. Let $R = 100$ cm, $\varepsilon = 0$, $\gamma = 1$ g/cm$^3$. Then $|\dot{\alpha}^{\circ}_{\max}| = 0.68 \times 10^{-4}$ 1/sec $= 0.0039^{\circ}$/sec. Calculations show that the stability effect of the solar pressure is very small for a cylinder whose radius equals 100 cm. However, if the radius is $R = 10$ cm, then $|\dot{\alpha}^{\circ}_{\max}| = 0.039^{\circ}$/sec, which represents already an appreciable magnitude.

Now we shall derive an expression for the forces and moments of light pressure, acting upon a body of arbitrary shape which changes its orientation relative to the light stream. For this purpose we shall introduce an "inertial" coordinate system $\xi\eta\zeta$ and a coordinate system $xyz$ centered in the body. The direction cosines of the $xyz$ system relative to the $\xi\eta\zeta$ system are functions of time. We denote the directions of the $x$, $y$, $z$ axes by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, respectively, and the direction of the exterior normal at every point of the surface of the body by $\mathbf{n}$

$$\mathbf{n} = \cos(\mathbf{n}, \mathbf{i})\,\mathbf{i} + \cos(\mathbf{n}, \mathbf{j})\,\mathbf{j} + \cos(\mathbf{n}, \mathbf{k})\,\mathbf{k} \tag{13}$$

Furthermore, we denote the direction opposite to that of the light flux by $\boldsymbol{\tau}$, and its direction cosines in the $x$, $y$, $z$ system by $a_0$, $b_0$, $c_0$, such that

$$\boldsymbol{\tau} = a_0\mathbf{i} + b_0\mathbf{j} + c_0\mathbf{k} \tag{14}$$

The quantities $a_0$, $b_0$, $c_0$ will also be functions of time.

The boundary of the illuminated part of the surface (terminator) shall be determined on the promise that at every point of the terminator the vector $\boldsymbol{\tau}$ lies in a tangent plane; consequently, at the boundary we have

$$\boldsymbol{\tau} \times \mathbf{n} = 0 \tag{15}$$

In the cases where the vector $\mathbf{n}$ is not a continuous function of the coordinates at some parts of the surface (cube, cylinder) one should utilize the relation

$$\boldsymbol{\tau} \times \mathbf{n} \geqslant 0 \tag{16}$$

for the determination of the terminator.

Note that the product $\boldsymbol{\tau} \times \mathbf{n}$ is nothing else but the cosine of the angle of incidence of the light stream (see (8) and (9)).

Further, we denote by $d\mathbf{f}$ an elemental force which is acting on an elemental area of the surface $dS$. Here

$$d\mathbf{f} = \mathbf{f} df = df\,(p\mathbf{i} + q\mathbf{j} + r\mathbf{k}) \tag{17}$$

where $\mathbf{f}$ is the unit vector of $d\mathbf{f}$ and $p$, $q$, $r$ are its direction cosines in the $x$, $y$, $z$ system.

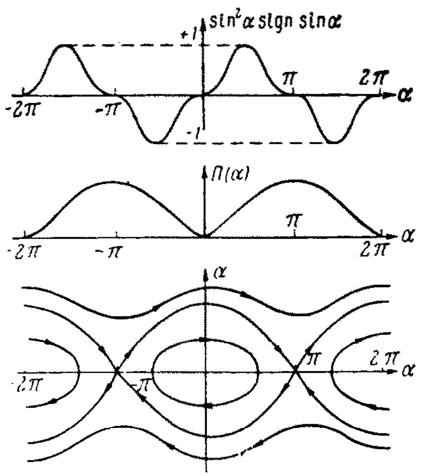


Fig. 3.

The magnitude of $df$ can be computed from (9). For the determination of $p$, $q$, $r$ three equations are necessary. One of them is the obvious relation

$$p^2 + q^2 + r^2 = 1 \tag{18}$$

In order to arrive at the second equation, we shall use the assumption that an incident ray and a reflected ray lie in the plane containing the normal of the area element. This means that the unit vectors $\mathbf{f}$, $\boldsymbol{\tau}$ and $\mathbf{n}$ lie in one plane, i.e.

$$(\mathbf{n} \times \boldsymbol{\tau}) \times \mathbf{f} = 0 \tag{19}$$

The third equation is obtained by substituting into (8) the values

$$\sin\vartheta = |\boldsymbol{\tau} \times \mathbf{n}|, \qquad \sin\varphi = |\mathbf{f} \times \mathbf{n}|$$

It can be easily shown that the solution of the system of Equations (18) and (8) obtained here has the form

$$\mathbf{f} = -\frac{(1-\varepsilon)\,\boldsymbol{\tau} + 2\varepsilon\mathbf{n}\,(\boldsymbol{\tau}\times\mathbf{n})}{\sqrt{(1-\varepsilon)^2 + 4\varepsilon^2\,(\boldsymbol{\tau}\times\mathbf{n})^2}} \tag{20}$$

From (20) it follows that in the case of a totally absorbing surface the direction of the force coincides with the direction of the light stream, i.e. opposite to the direction of $\boldsymbol{\tau}$ and

$$p = -a_0, \qquad q = -b_0, \qquad r = -c_0 \tag{21}$$

In the case of a totally reflecting surface the direction of the force is opposite to the direction of the exterior normal $\mathbf{n}$ and

$$p = -\cos(\mathbf{n}, \mathbf{i}), \qquad q = -\cos(\mathbf{n}, \mathbf{j}), \qquad r = -\cos(\mathbf{n}, \mathbf{k}) \qquad (22)$$

The total force $\mathbf{F}$ of the light pressure, which is acting on a body of arbitrary shape is equal to

$$\mathbf{F} = \int_{(S_1)} d\mathbf{f}$$

where $S_1$ is the illuminated part of the surface, whose boundary is determined from (16).

If we consider (21) and (22) and also the above remark on the equality of the cosine of the incidence angle with $\boldsymbol{\tau} \times \mathbf{n}$ we obtain an expression for a force $F^+$ which acts on a body bounded by a totally absorbing surface, and for a force $F^-$ which acts on a body bounded by a totally reflecting surface

$$\mathbf{F}^+ = -h_0\boldsymbol{\tau} \int_{(S_1)} (\boldsymbol{\tau}\times\mathbf{n})\, dS, \qquad \mathbf{F}^- = -2h_0 \int_{(S_1)} \mathbf{n}\,(\boldsymbol{\tau}\times\mathbf{n})^2\, dS \quad \left(h_0 = \frac{E_0}{c}\left(\frac{r_0}{r}\right)^2\right) \quad (23)$$

Here the integration is carried out over the region $\boldsymbol{\tau} \times \mathbf{n} \geqslant 0$.

Using Equations (9), (20) and (23) we find that the force acting on the body bounded by a surface with an arbitrary reflection coefficient is given by the formula

$$\mathbf{F} = (1-\varepsilon)\,\mathbf{F}^+ + \varepsilon\mathbf{F}^- = \mathbf{F}^+ + \varepsilon\,(\mathbf{F}^- - \mathbf{F}^+)$$

Let $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the radius vector of a point on the surface in question, where the coordinates $x$, $y$, $z$ are related by the surface equation. Then the elemental moment of light pressure will equal $d\mathbf{M} = \mathbf{R} \times d\mathbf{f}$ and the total moment will be

$$\mathbf{M} = \int_{(S_1)} \mathbf{R} \times d\mathbf{f}$$

From the foregoing it follows that it is possible to obtain rather simply expressions for the moment of light pressure acting on a body with a completely absorbing surface $\mathbf{M}^+$ and on a body with a completely reflecting surface $\mathbf{M}^-$

$$\mathbf{M}^+ = h_0\boldsymbol{\tau} \times \int_{(S_1)} \mathbf{R}\,(\boldsymbol{\tau}\times\mathbf{n})\, dS, \qquad \mathbf{M}^- = 2h_0 \int_{(S_1)} \mathbf{n} \times \mathbf{R}\,(\boldsymbol{\tau}\times\mathbf{n})^2\, dS \qquad (24)$$

and for a body bounded by a surface with an arbitrary reflection coefficient

$$\mathbf{M} = (1-\varepsilon)\,\mathbf{M}^{+} + \varepsilon\,\mathbf{M}^{-} = \mathbf{M}^{+} + \varepsilon\,(\mathbf{M}^{-} - \mathbf{M}^{+})$$

The combination of Formulas (23), (24) and (16) gives the force action of the light stream on a body placed in that stream. The computation of the forces and moments of light pressure by means of (23) and (24) is rather complicated even for simple surfaces since the integrand as well as the integration limits depend on both the surface parameters and on the orientation of the light stream relative to the body. Thus it is desirable to simplify the above expressions so as to ease the calculations. It turns out that this can be done easily for $\mathbf{F}^{+}$ and $\mathbf{M}^{+}$.

For this purpose we add the illuminated part of the surface to the closed surface consisting of the illuminated part of the surface of the body, the lateral cylindrical surface whose generator is parallel to the direction of the light stream and whose direction is along the terminator (Fig. 4), and the flat bottom, perpendicular to the direction of the light stream. We shall apply to this closed surface the Ostrogradskii-Gauss formula. As a result we obtain

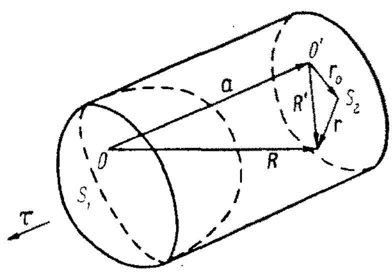

Fig. 4.

$$\mathbf{F}^{+} = h_0\boldsymbol{\tau}S_2, \qquad \mathbf{M}^{+} = h_0\boldsymbol{\tau} \times \int_{(S_2)} \mathbf{R}\,dS \tag{25}$$

where $S_2$ is the projection of the illuminated part of the surface onto a plane perpendicular to the light stream.

We denote the point, relative to which the moment is calculated, by $O$, and the projection of this point onto a plane (Fig. 4) perpendicular to the vector $\boldsymbol{\tau}$, by $O'$. After changing the vector $\mathbf{R}$ to the sum $\mathbf{R} = \mathbf{a} + \mathbf{R}'$ we obtain

$$\mathbf{M}^{+} = h_0\boldsymbol{\tau} \times \int_{(S_2)} \mathbf{R}'dS \tag{26}$$

where the vector $\mathbf{R}'$ is the radius-vector of a point in the region $S_2$ relative to the point $O'$.

Furthermore, we represent the vector $\mathbf{R}'$ as the sum $\mathbf{R}' = \mathbf{r} + \mathbf{r}_0$, where $\mathbf{r}_0$ is the radius vector of the center of gravity of the bottom relative to the origin of the radius vector $\mathbf{R}'$, and $\mathbf{r}$ is the running radius vector of a point relative to the center of gravity of the bottom.

After substituting $\mathbf{R}'$ into (26) we obtain

$$\mathbf{M}^{+} = h_0 S_2 \boldsymbol{\tau} \times \mathbf{r}_0 \tag{27}$$

Here $\mathbf{r}_0$ is the radius vector of the center of gravity of the projection of the illuminated part of the surface onto the $S_2$ plane with respect to that point relative to which the moment on the plane is calculated.

In regard to $\mathbf{F}^{-}$ and $\mathbf{M}^{-}$, however, it was not possible to obtain significant improvements of the calculation method. These quantities have to be determined by straight integration.

As an example we introduce results of calculations of forces and moments due to light pressure which act on bodies bounded by various definite surfaces.

*1. Ellipsoid*

$$x^2 / A^2 + y^2 / B^2 + z^2 / C^2 = 1$$

In the case of a totally absorbing surface

$$\mathbf{F}^{+} = -h_0 \boldsymbol{\tau} \pi \sqrt{a_0^2 B^2 C^2 + b_0^2 A^2 C^2 + c_0^2 A^2 B^2} \tag{28}$$

where $a_0$, $b_0$, $c_0$ are the direction cosines of the vector $\boldsymbol{\tau}$ relative to the $x$, $y$, $z$ axes, respectively. The moment with respect to the center of the ellipsoid, which coincides with the center of mass, is equal to zero.

*2. Right circular cylinder*

$$x^2 + y^2 = R^2, \quad |z| \leqslant a$$

In the case of a totally absorbing surface

$$\mathbf{F}^{+} = -h_0 \boldsymbol{\tau} \pi R^2 \left[ |c_0| + \frac{4}{\pi} \frac{a}{R} \sqrt{1 - c_0^2} \right] \tag{29}$$

The moment with respect to the center of the cylinder is equal to zero.

*3. Right circular cone*

$$x^2 + y^2 = R^2 \left(1 - \frac{z}{h}\right)^2, \qquad 0 \leqslant z \leqslant h$$

In the case of a totally absorbing surface

$$\mathbf{F}^{+} = -h_0 \tau \pi R^2 |c_0| \qquad (\Omega \leqslant 0) \tag{30}$$

$$\mathbf{F}^{+} = -h_0 \tau \pi R^2 \left\{ \left[ 1 - \frac{f(|c_0|)}{2\pi} \right] |c_0| + \frac{h}{\pi R} \Omega^{\frac{3}{2}} \sqrt{1 - c_0^2} \right\} \qquad (\Omega \geqslant 0)$$

Here

$$\Omega = 1 - \left(\frac{R}{h}\right)^2 \frac{c_0^2}{1 - c_0^2}$$

$$f(|c_0|) = \sin^{-1}\left[2\left(\frac{R}{h}\right)\frac{|c_0|}{\sqrt{1-c_0^2}}\Omega^{\frac{1}{2}}\right] - 2\left(\frac{R}{h}\right)\frac{|c_0|}{\sqrt{1-c_0^2}}\Omega^{\frac{1}{2}}$$

$$f(|c_0|) = 0 \quad \text{for } \Omega = 0, \quad f(|c_0|) = \pi \quad \text{for } c_0 - 0$$

The projections of the moment with respect to the center of mass onto the $x'$, $y'$, $z$ axes (axes $x'$, $y'$ are parallel to $x$, $y$ but pass through the center of mass of the cone) are equal to:

for $\Omega \leqslant 0$

$$M_{x'} = -\frac{1}{4} h_0 b_0 h\pi \quad R^2 | c_0, \quad |M_{y'} = \frac{1}{4} h_0 a_0 h\pi \quad R^2 | c_0 |, \; M_z = 0 \tag{31}$$

for $\Omega \geqslant 0$

$$M_{x'} = -\frac{1}{4} h_0 b_0 h\pi R^2 \left\{\left[1 - \frac{f(|c_0|)}{2\pi}\right] | c_0 | - \frac{1}{3}\frac{h}{R}\Omega\sqrt{1 - c_0^2}\right\}$$

$$M_{y'} = \frac{1}{4} h_0 a_0 h\pi R^2 \left\{\left[1 - \frac{f(|c_0|)}{2\pi}\right] | c_0 | - \frac{1}{3}\frac{h}{R}\Omega\sqrt{1 - c_0^2}\right\}$$

$$M_z = 0$$

*4. Hemisphere*

$$x^2 + y^2 + z^2 = R^2, \qquad z \geqslant 0$$

In the case of a totally absorbing surface

$$\mathbf{F}^+ = -h_0\tau \frac{\pi R^2}{2} (1 + | c_0 |) \tag{32}$$

The projections of the moment with respect to the center of mass of the hemisphere onto the axes $x'$, $y'$, $z$ (axes $x'$, $y'$ are parallel to $x$, $y$ but pass through the center of mass of the hemisphere) are equal to:

$$M_{x'} = -h_0 b_0 \frac{3}{16} \pi R^3\left[1 + | c_0 | - \frac{32}{9\pi}\sqrt{1 - c_0^2}\right],$$

$$M_{y'} = h_0 a_0 \frac{3}{16}\pi R^3\left[1 + | c_0 | - \frac{32}{9\pi}\sqrt{1 - c_0^2}\right], \qquad M_z = 0 \tag{33}$$

It can be seen from (33) that the moment does not approach zero at $c_0 = 0$ as would be expected, (the same takes place in the previous example). This is explained by the fact that the centers of gravity of the body of revolution and of the section of this body by an axial plane do not coincide in general.

As was already pointed out, the calculation of forces and moments due to light pressure acting on a body bounded by a totally reflecting surface is considerably more complicated than in the case of the totally absorbing surface, and the results are more complex and cumbersome relations. Thus in the case of a completely reflecting surface we shall restrict ourselves only to the surfaces of a sphere and a right circular cylinder.

5. *Sphere* in the case of a totally reflecting surface

$$\mathbf{F}^- = -h_0\tau\pi R^2, \qquad \mathbf{M}^- = 0 \tag{34}$$

If we let in (28) $A = B = C = R$, i.e. examine a particular case of an ellipsoid-sphere, then it can be seen that the force action of the light flux onto the sphere is the same for totally absorbing and for totally reflecting surfaces.

6. *Right circular cylinder* in the case of a completely reflecting surface

$$F^-_x = -\frac{16}{3} h_0 a_0 R a \sqrt{1 - c_0^2}, \qquad F_y^- = -\frac{16}{3} h_0 b_0 R a \sqrt{1 - c_0^2}$$

$$F_z^- = -2h_0\pi R^2 c_0^2 \operatorname{sign} c_0, \qquad \mathbf{M}^- = 0 \tag{35}$$

7. *Body of revolution*

$$x^2 + y^2 = f(z) \qquad (z \text{ is the axis of revolution}) \tag{36}$$

Let us study the case of a completely absorbing surface.

If the surface of revolution is also bounded by a face plane, then one has to add to Equation (36) also the inequality

$$z \leqslant z^* \tag{37}$$

where $z^*$ is a bounding quantity. Let us denote, as earlier, the unit vectors of the axes $x$, $y$, $z$ by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, respectively and determine the direction cosines of the exterior normal at every point of the lateral surface

$$\cos(\mathbf{n}, \mathbf{i}) = \frac{x}{\sqrt{f(z) + {}^1/_4 [f'(z)]^2}}, \quad \cos(\mathbf{n}, \mathbf{j}) = \frac{y}{\sqrt{f(z) + {}^1/_4 [f'(z)]^2}} \tag{38}$$

$$\cos(\mathbf{n}, \mathbf{k}) = -\frac{{}^1/_2 f'(z)}{\sqrt{f(z) + {}^1/_4 [f'(z)]^2}} \qquad \left(f'(z) = \frac{df}{dz}\right)$$

After substituting (38) into the equation of the terminator $\tau \times \mathbf{n} = 0$ we obtain

$$a_0 x + b_0 y = \frac{1}{2} f'(z) c_0 \tag{39}$$

On the end surfaces

$$\cos(\mathbf{n}, \mathbf{i}) = 0, \quad \cos(\mathbf{n}, \mathbf{j}) = 0, \quad \cos(\mathbf{n}, \mathbf{k}) = \mp 1$$

Now we introduce a new coordinate system $x''$, $y''$, $z''$, formed from the old one by rotations $\beta$ and $\vartheta$ so that the table of direction cosines of the new coordinate system has the following appearance with respect to the old system (Fig. 5):

| | $x$ | $y$ | $z$ |
|---|---|---|---|
| $x''$ | $\cos\beta$ | $\sin\beta$ | 0 |
| $y''$ | $-\sin\cos\vartheta$ | $\cos\beta\cos\vartheta$ | $\sin\vartheta$ |
| $z''$ | $\sin\beta\sin\vartheta$ | $-\cos\beta\sin\vartheta$ | $\cos\vartheta$ |

Here

$$\cos\beta = \frac{b_0}{\sqrt{1-c_0^2}}, \quad \sin\beta = -\frac{a_0}{\sqrt{1-c_0^2}}, \quad \cos\vartheta = \sqrt{1-c_0^2}, \quad \sin\vartheta = c_0$$

$$\left(0 \leqslant \beta \leqslant 2\pi, \ -\frac{\pi}{2} \leqslant \vartheta \leqslant \frac{\pi}{2}\right)$$

We denote the unit vectors of the $x''$, $y''$, $z''$ axes by $\mathbf{i}''$, $\mathbf{j}''$, $\mathbf{k}''$, respectively. Then it is readily established that $\mathbf{j}'' \times \boldsymbol{\tau} = 1$ and $\mathbf{k} \times \boldsymbol{\tau} = 0$, i.e. the plane $x''z''$ is perpendicular to the direction of the light stream, and the vector $\mathbf{k}$ lies in the $z''y''$-plane.

After replacing the old coordinates in (36) and (39) by the new ones we obtain

$$x''^2 + \left[y''\sqrt{1-c_0^2} - z''c_0\right]^2 = f(z) \text{ for } z = y''c_0 + z''\sqrt{1-c_0^2} \tag{40}$$

$$\left[y''\sqrt{1-c_0^2} - z''c_0\right]\sqrt{1-c_0^2} = \frac{1}{2}f'(z) \quad \text{for } z = y''c_0 + z''\sqrt{1-c_0^2}$$

If we eliminate the $y''$ coordinate from (40), we obtain a coupling equation between $x''$ and $z''$, i.e. the equation of the projection of the terminator onto a plane perpendicular to the light stream. For the calculation of forces and moments acting on a body with a totally absorbing surface it is necessary to determine the magnitude of the area of this projection. One can arrive at the following conclusions on the basis of the general form of Equations (40): (1) The magnitude of this area depends only on the parameters of the surface itself, and the angle between the axis of body symmetry and the direction of the light stream; (2) the curve bounding the projection is symmetric relative to the $z''$-axis, i.e. the center of gravity of the projection area lies on the $z''$-axis.

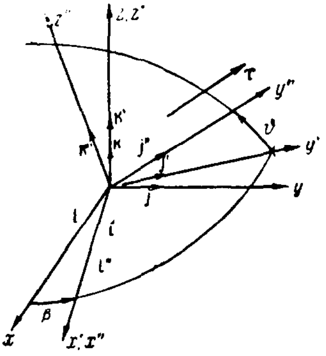


Fig. 5.

In the derivation of the Equations (40) it was assumed that the terminator lies completely on the lateral surface of the body and does not pass over the end face. In order to take into account the influence of the end face it is necessary to obtain the condition for the intersection of the terminator with the plane of the face. This condition can be written as follows:

$$2R \geqslant f'(a) \sqrt{\frac{c_0^2}{1-c_0^2}} \tag{41}$$

where $R$ is the radius of the face and $a$ is the distance of the face from the coordinate origin. When this condition is satisfied and we replace the old variables by new ones in the equation $x^2 + y^2 = R^2$ and $z = a$, the equation of the projection of the end-face onto the $x''z''$-plane becomes

$$c_0^2 x''^2 + [a\sqrt{1-c_0^2} - z'']^2 = c_0^2 R^2 \tag{42}$$

The intersection of this curve with the curve defined by Equation (40) presents the complete projection of the illuminated part of the surface onto a plane which is perpendicular to the light stream. As can be seen, the consideration of the influence of the end face does not change the general conclusions made earlier. Consequently, the area of the projection can be considered to be a function of the direction cosine $c_0$, i.e. $S_2 = S_2(c_0)$.

Considering that the center of gravity of the projection area lies on the $z''$-axis, we can write the vector $\mathbf{r}_0$ in the form $\mathbf{r}_0 = r_0\mathbf{k}''$ and the moment acting on the body of revolution with a totally absorbing surface in the form

$$\mathbf{M}^+ = h_0 S_2(c_0)\, r_0\, [\boldsymbol{\tau} \times \mathbf{k}''] \tag{43}$$

The magnitude $r_0$ of the vector $\mathbf{r}_0$ is also a function of the surface parameters and the direction cosine $c_0$, i.e. $r_0 = r_0(c_0)$. Using the table of direction cosines we find the following projections onto the respective axes:

$$[\boldsymbol{\tau} \times \mathbf{k}'']_x = \frac{b_0}{\sqrt{1-c_0^2}}, \quad [\boldsymbol{\tau} \times \mathbf{k}'']_y = -\frac{a_0}{\sqrt{1-c_0^2}}, \quad [\boldsymbol{\tau} \times \mathbf{k}'']_z = 0 \tag{44}$$

Then we represent Equation (43) with consideration of (44) in terms of the projections onto the axes as follows:

$$M_x^+ = \frac{b_0 h_0}{\sqrt{1-c_0^2}} S_2(c_0)\, r_0(c_0), \qquad M_y^+ = -\frac{a_0 h_0}{\sqrt{1-c_0^2}} S_2(c_0)\, r_0(c_0)$$

$$M_z^+ = 0 \tag{45}$$

The magnitude of the total moment due to light pressure will be equal to

$$|\mathbf{M}^+| = \sqrt{(M_x^+)^2 + (M_y^+)^2} = h_0 S_2(c_0) r_0(c_0) \tag{46}$$

It follows from (46) that $|\mathbf{M}^+| = 0$ for $c_0 = \mp 1$, since for $c_0 = \mp 1$ the projection of the illuminated part of the surface is a circle with its center at the axis of symmetry. However, $M_x^+$ and $M_y^+$ can approach zero not only for $c_0 = \pm 1$, but also for some values of $c_0 = c_0^*$, when $r_0(c_0^*) = 0$.

If it becomes necessary during the determination of the moment to go from point $O$, relative to which the moment is calculated by (45), to the point $C$, which lies on the axis of the body at a distance $\sigma_z$ from point $O$, then

$$M_{x'}^+(C) = \frac{b_0 h_0 S_2(c_0)}{\sqrt{1-c_0^2}} [r_0(c_0) + \sigma_z \sqrt{1-c_0^2}]', \qquad M_z^+(C) = 0 \tag{47}$$
$$M_{y'}^+(C) = -\frac{a_0 h_0 S_2(c_0)}{\sqrt{1-c_0^2}} [r_0(c_0) + \sigma_z \sqrt{1-c_0^2}]'$$

As before, $M_{x'}^+(C) = M_{y'}(C) = 0$ for $|c_0| = 1$, $a_0 = b_0 = 0$. It may turn out, however, that according to (46) these components will be equal to zero for some value $c_0^*$, which is a root of Equation

$$r_0(c_0) + \sigma_z \sqrt{1-c_0^2} = 0 \tag{48}$$

For a body of revolution in the case of a totally reflecting surface, whose equation is written in terms of the $x$, $y$, $z$ axes having their origin at the point $O$, the projections of the moment onto the $x'$, $y'$, $z$ axes having their origin at the point $C$, will be

$$M_{x'}^-(C) = 2h_0 \int_{(S_1)} (\boldsymbol{\tau}\times\mathbf{n})^2 y \frac{z + \frac{1}{2} f'(z) + \sigma_z}{\sqrt{f(z) + \frac{1}{4}[f'(z)]^2}} \, dS,$$
$$M_{y'}^-(C) = -2h_0 \int_{(S_1)} (\boldsymbol{\tau}\times\mathbf{n})^2 x \frac{z + \frac{1}{2} f'(z) + \sigma_z}{\sqrt{f(z) + \frac{1}{4}[f'(z)]^2}} \, dS, \qquad M_z^-(C) = 0 \tag{49}$$

where $\sigma_z$ is the distance between the points $O$ and $C$.

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